Lecture 12 16-10-18

Recall:

We have

$$\left(\frac{d}{dt}\right)^k \left(e^{\lambda t} y\right) = e^{\lambda t}\left[\left(\frac{d}{dt}+\lambda\right)^k y\right]$$

using the computation:

$$\frac{d}{dt}\left(e^f y\right) = e^f\left(\frac{dy}{dt} + \frac{df}{dt} y\right).$$

for general function f.

- Let $a_n y^{(n)} + \cdots + a_0 y = 0$ with

Let $f(r) = a_n (r-r_1) \cdots (r-r_s)(r-r_0)^{n-s}$

Last time: we show $f\left(\frac{d}{dt}\right)\left(t^k e^{r_0 t}\right) = 0$

for $k < n-s$.

$\Longrightarrow$ $e^{r_0 t}, te^{r_0 t}, \ldots, t^{n-s-1} e^{r_0 t}$ are solutions

to the ODE.

Fundamental set:

$$e^{r_1 t}, e^{r_2 t}, \ldots, e^{r_s t}, e^{r_0 t}, te^{r_0 t}, \ldots, t^{n-s-1} e^{r_0 t}.$$

Next: we consider

$$f(r) = a_n(r-r_1)\cdots(r-r_s)(r-\lambda)^k(r-\bar{\lambda})^k$$

with $2k+s=n$.

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Claim: $e^{\lambda t}, \ldots, t^{k-1}e^{\lambda t}$

$e^{\bar{\lambda}t}, \ldots, t^{k-1}e^{\bar{\lambda}t}$ are solutions

Reason: if we write

$$f\left(\frac{d}{dt}\right) = a_n\left(\frac{d}{dt}-r_1\right)\cdots\left(\frac{d}{dt}-r_s\right)\left(\frac{d}{dt}-\lambda\right)^k\left(\frac{d}{dt}-\bar{\lambda}\right)^k$$

then $a_n y^{(n)} + \cdots + a_1 y^{(1)} + a_0 y = f\left(\frac{d}{dt}\right)(y)$ ← operator.

Observe: $\underbrace{\left(e^{\lambda t}\frac{d}{dt}e^{-\lambda t}\right)}_{\text{operator!}}(y) = \left(\frac{d}{dt}-\lambda\right)(y)$

∴ we can rewrite

$$\left(\frac{d}{dt}-\lambda\right)^k = \left(e^{\lambda t}\frac{d}{dt}e^{-\lambda t}\right)^k$$

$$= \left(e^{\lambda t}\frac{d}{dt}\cancel{e^{-\lambda t}}\right)\left(\cancel{e^{\lambda t}}\frac{d}{dt}e^{-\lambda t}\right)\cdots\left(e^{\lambda t}\frac{d}{dt}e^{-\lambda t}\right)$$

$$= e^{\lambda t}\left(\frac{d}{dt}\right)^k e^{-\lambda t}$$

$$\therefore \left(\frac{d}{dt}-\lambda\right)^k\left(t^m e^{\lambda t}\right) = e^{\lambda t}\left(\frac{d}{dt}\right)^k(t^m)$$

$= 0$ if $m < k$

Similarily: $\left(\frac{d}{dt} - \bar{\lambda}\right)^k = e^{\bar{\lambda}t}\left(\frac{d}{dt}\right)^k e^{-\bar{\lambda}t}$

$$\left(\frac{d}{dt} - \bar{\lambda}\right)^k \left(e^{\bar{\lambda}t} t^m\right) = e^{\bar{\lambda}t}\left(\frac{d}{dt}\right)^k (t^m) = 0$$

if $m < k$.

To obtain $\mathbb{R}$-valued solution: write $\lambda = \alpha + i\mu$

$$e^{\alpha t}\cos\mu t = \left(\frac{e^{\lambda t} + e^{\bar{\lambda}t}}{2}\right) \quad \dots \quad t^{k-1} e^{\alpha t}\cos\mu t$$

$$e^{\alpha t}\sin\mu t = \left(\frac{e^{\lambda t} - e^{\bar{\lambda}t}}{2i}\right) \quad \dots \quad t^{k-1} e^{\alpha t}\sin\mu t.$$

Example: Let $f(r) = (r^2+1)(r-1)^2(r+2)$

then we consider solution for $f\left(\frac{d}{dt}\right)(y) = 0$

We have: $r_1 = 1$ with multiplicity 2

$r_2 = -2$ " " 1.

$\lambda = i$, $\bar{\lambda} = -i$ " " 1.

$\therefore \quad y_1 = e^t, \; y_2 = e^{-2t}, \; y_3 = \cos t, \; y_4 = \sin t.$

Ex: Compute the Wronskian and show that they are fundamental set of sol.

Example 2: $f(r) = (r^2+2r+2)^2(r-1)^3$

Consider the equation: $f(\frac{d}{dt})(y) = 0$

We first solve $f(r)=0$, which gives us

$r_1 = 1$ with multiplicity $=3$

$\lambda = -1+i$, $\bar{\lambda} = -1-i$ multi $=1$

Hence we have

$y_1(t) = e^t$, $y_2 = te^t$, $y_3 = t^2e^t$

$y_4(t) = e^{-t}\cos t$, $y_5 = e^{-t}\sin t$

Recall: Among all these examples, we have to prove that $y_1, \dots, y_n$ is a fundamental set.

Thm: $y_1, \dots, y_n$ solution to $y^{(n)} + p_{n-1}(t)y^{(n-1)} + \dots + p_0(t)y = 0$

TFAE:
1) $y_1, \dots, y_n$ linearly independent.
2) $\text{Span}(y_1, \dots, y_n) = \mathcal{S}$
3) $W(y_1, \dots, y_n)(t_0) \neq 0$ for some $t_0 \in I$.

Eg. $f(r) = a_n (r-r_1)^2 (r-r_2)^3$ s.t. $2+3=n$

$\rightsquigarrow \quad e^{r_1 t}, te^{r_1 t}, e^{r_2 t}, te^{r_2 t}, t^2 e^{r_2 t}$

Want: they are Linearily independent.

Assume:

$$0 = c_1 e^{r_1 t} + c_2 t e^{r_1 t} + c_3 e^{r_2 t} + c_4 t e^{r_2 t} + c_5 t^2 e^{r_2 t}$$

Want: $c_i = 0$ for $i = 1, \dots, 5$.

Try: $0 = c_1 + c_2 t + c_3 e^{\hat{r}t} + c_4 t e^{\hat{r}t} + c_5 t^2 e^{\hat{r}t}$.

with $\hat{r} = r_2 - r_1$

differentiate:

$$c_2 + (\hat{r}c_3 + c_4)e^{\hat{r}t} + (\hat{r}c_4 + 2c_5) t e^{\hat{r}t} + \hat{r} c_5 t^2 e^{\hat{r}t}$$

Need: $1, e^{\hat{r}t}, te^{\hat{r}t}, t^2 e^{\hat{r}t}$ linearily indep.

True: by induction

$\Longrightarrow \quad c_2 = 0, \ \hat{r}c_3 + c_4 = 0, \ \hat{r}c_4 + 2c_5 = 0, \ \hat{r}c_5 = 0$

$\Rightarrow \quad c_2 = c_3 = c_4 = c_5 = 0$

Back to original eqt: $\Longrightarrow \quad c_1 = 0$

E.g. More generally:

$$f(r) = a_n (r-r_1)^2 (r-\lambda)^2 (r-\bar{\lambda})^2.$$

$\lambda = \alpha + i\mu.$

we have

$e^{r_1 t}$, $te^{r_1 t}$, $e^{\alpha t}\cos\mu t$, $e^{\alpha t}\sin\mu t$, $te^{\alpha t}\cos\mu t$, $te^{\alpha t}\sin\mu t$.

Want: They are linearly independent!

First: we show

$e^{r_1 t}$, $te^{r_1 t}$, $e^{\lambda t}$, $te^{\lambda t}$, $e^{\bar{\lambda} t}$, $te^{\bar{\lambda} t}$

linearly independent over $\mathbb{C}$

- same induction proof as previous example.

then: writing $e^{\alpha t}\cos\mu t = \left(\dfrac{e^{\lambda t} + e^{\bar{\lambda} t}}{2}\right)$

$$e^{\alpha t}\sin\mu t = \left(\frac{e^{\lambda t} - e^{\bar{\lambda} t}}{2i}\right)$$

$\Rightarrow$ $e^{r_1 t}$, $te^{r_1 t}$, $e^{\alpha t}\cos\mu t$, $e^{\alpha t}\sin\mu t$, $te^{\alpha t}\cos\mu t$, $te^{\alpha t}\sin\mu t$

linearly independent over $\mathbb{C}$ $\Rightarrow$ l.i over $\mathbb{R}$

§ Inhomogeneous equation with constant coefficient

$$(*) \text{------} \quad a_n y^{(n)} + \cdots + a_1 y + a_0 y = r(t).$$

particular solution

Prop: General solution:

$$c_1 y_1 + \cdots + c_n y_n + Y(t)$$

solutions to the homogeneous equation!

<u>Goal:</u> to find $Y(t)$

<u>Three cases:</u>

<u>Case i)</u> $r(t) = P_k(t)$

<u>Case ii)</u> $r(t) = e^{\alpha t} P_k(t)$

<u>Case iii)</u> $r(t) = e^{\alpha t} P_k(t) \cos\mu t$ or $e^{\alpha t} P_k(t) \sin\mu t$.

<u>Case i)</u> Write $P_k(t) = A_0 t \cdots + A_k t^k$.

And let $Y(t) = Q_\ell(t) = B_0 t + \cdots + B_\ell t^\ell$

<u>Want:</u> Solve for B_i's.

<u>Ex:</u> $a_3 y^{(3)} + a_2 y^{(2)} + a_1 y' + a_0 y = P_k(t)$.

$$\rightsquigarrow \quad Y' = B_1 + 2B_2 t + 3B_3 t^2 + \dots + (j+1)B_j t^j + \dots + \ell B_\ell t^{\ell-1}$$

$$Y'' = 2B_2 + 3\cdot 2\, B_3 t + \dots + (j+2)(j+1) B_{j+2} t^j + \dots + \ell(\ell-1) B_\ell t^{\ell-2}$$

$$Y^{(3)} = 3\cdot 2 B_3 + \dots + (j+3)(j+2)(j+1) B_{j+3} t^j + \dots + \ell(\ell-1)(\ell-2) B_\ell t^{\ell-3}$$

$\therefore$ coefficient of t^j in $f(\frac{d}{dt})(Y)$:

$$a_3 \frac{(j+3)!}{j!} B_{j+3} + a_2 \frac{(j+2)!}{j!} B_{j+2} + a_1 \frac{(j+1)!}{j!} B_{j+1} + a_0 \frac{j!}{j!} B_j$$

Equation to solve:

$$a_3 \frac{(j+3)!}{j!} B_{j+3} + a_2 \frac{(j+2)!}{j!} B_{j+2} + a_1 \frac{(j+1)!}{j!} B_{j+1} + a_0 \frac{j!}{j!} B_j = A_j .$$

4 cases:

1) $a_0 \neq 0$, then we can solve it with $\ell = k$:

invertible

$$\begin{pmatrix} a_0 & a_1 & 2a_2 & \cdots & - \\ & \ddots & \ddots & \ddots & \vdots \\ & & \ddots & \ddots & a_2 \frac{k!}{(k-2)!} \\ & 0 & & \ddots & a_1 k \\ & & & & a_0 \end{pmatrix} \begin{pmatrix} B_0 \\ \vdots \\ \vdots \\ B_k \end{pmatrix} = \begin{pmatrix} A_0 \\ \vdots \\ \vdots \\ A_k \end{pmatrix}$$

2) $a_0 = 0$, $a_1 \neq 0$, we can solve it with $l = k+1$.

$$\begin{pmatrix} a_1 & & * \\ & \ddots & \\ 0 & & a_1(k+1) \end{pmatrix} \begin{pmatrix} B_1 \\ \vdots \\ B_{k+1} \end{pmatrix} = \begin{pmatrix} A_0 \\ \vdots \\ A_k \end{pmatrix}$$

3) $a_0 = a_1 = 0$, $a_2 \neq 0$, take $l = k+2$

$$\begin{pmatrix} 2a_2 & & * \\ & \ddots & \\ 0 & & a_2 \frac{(k+2)!}{k!} \end{pmatrix} \begin{pmatrix} B_2 \\ \vdots \\ B_{k+2} \end{pmatrix} = \begin{pmatrix} A_0 \\ \vdots \\ A_k \end{pmatrix}$$

4) $a_0 = a_1 = a_2 = 0$, we take $l = k+3$,

$$\begin{pmatrix} 6a_3 & & * \\ & \ddots & \\ 0 & & a_3 \frac{(k+3)!}{k!} \end{pmatrix} \begin{pmatrix} B_3 \\ \vdots \\ B_{k+3} \end{pmatrix} = \begin{pmatrix} A_0 \\ \vdots \\ A_k \end{pmatrix}$$

In general:

For $f(r) = a_n r^n + \dots + a_0$

and consider $f(\frac{d}{dt})(Y) = P_k(t)$.

General equation:

$$\boxed{a_n \frac{(j+n)!}{j!} B_{j+n} + a_{n-1} \frac{(j+n-1)!}{j!} B_{j+n-1} + \dots + a_0 \frac{j!}{j!} B_j = A_j}$$

n Cases: $a_0 = \dots = a_m = 0, \quad a_{m+1} \neq 0$

we can solve it will $l = k+m+1$

- Next we consider $r(t) = e^{\alpha t} P_k(t) = A_0 + \dots + A_k t^k$

Try: $Y(t) = e^{\alpha t} Q_l(t) = B_0 + \dots + B_l t^l$